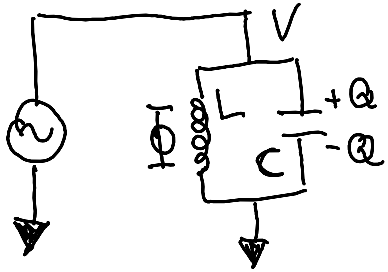
Electronic Harmonic Oscillator



- charge on capacitor

$$Q = CV$$

- flux in the inductor

$$\Phi = LI$$

- voltage across oscillator

$$V = \frac{Q}{C} = -L\dot{I} = -\dot{\Phi}$$

Hamiltonian

$$H = \frac{CV^2}{2} + \frac{LI^2}{2} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

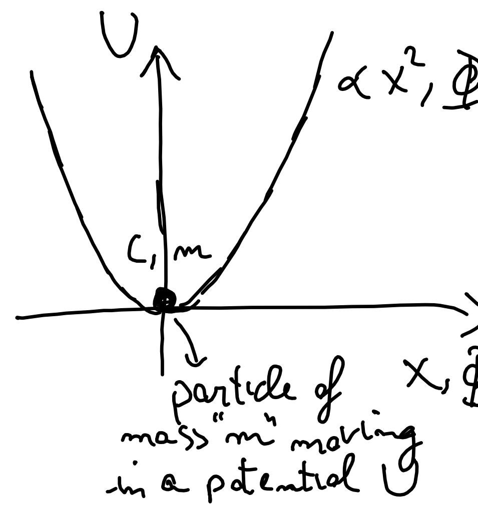
electrostatic energy

magnetic energy

- Compare to mechanical harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

kinetic + potential



Characteristic quantities

mechanical	electronic
position x	flux Φ
momentum p	charge Q
mass m	capacitance C
spring constant K	inverse inductance $1/L$
resonance freq. $\omega = \sqrt{\frac{k}{m}}$	$\omega = \sqrt{\frac{1}{LC}}$

Harmonic oscillator

conjugate variables

$$\frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I = \dot{Q}$$

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = V = -L\dot{I} = -\dot{\Phi}$$

$$\boxed{\begin{array}{l} x, p \\ \Phi, Q \end{array} \text{ are canonical variables}}$$

- quantum mechanical operators

$$\hat{x} = x \qquad \hat{\Phi} = \Phi$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \qquad \hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$$

- commutation relation

$$[\hat{x}, \hat{p}] = i\hbar \qquad [\hat{\Phi}, \hat{Q}] = i\hbar \Leftrightarrow \left[2\pi\frac{\hat{\Phi}}{\Phi_0}, \frac{Q}{2e}\right] = [\hat{\delta}, \hat{N}] = i$$

Hamiltonian operator

- using conjugate variables Q, Φ

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial \Phi^2} + \frac{1}{2L}\Phi^2$$

- using creation and annihilation operators

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q}^+ - i\hat{\Phi}^+\right) \quad \text{creation}$$

operators

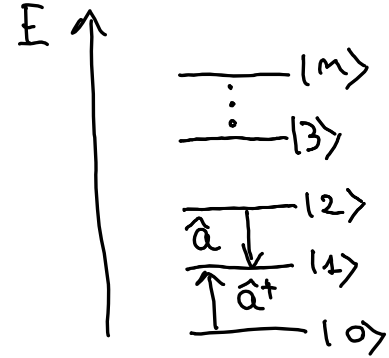
$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q} + i\hat{\Phi}\right) \quad \text{annihilation}$$

with $Z_c = \sqrt{L/C}$ impedance of oscillator

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

$a^+ a = n$ number operators

- Properties of $\hat{a}^+$ and $\hat{a}$



Spectrum

$$\hat{a}^+ |n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$\hat{a}\ |n\rangle = \sqrt{n}\ |n-1\rangle$$

$$\hat{a}^+ \hat{a} |n\rangle = n\, |n\rangle$$

with $|n\rangle$ number (Fock) state of harmonic oscillator

- relation to $\hat{Q}$ and $\hat{\Phi}$

$$\hat{Q} = \sqrt{\frac{\hbar}{2 Z_c}}\ (\hat{a}^+ + \hat{a}) \longrightarrow$$ related to electric field stored on capacitor

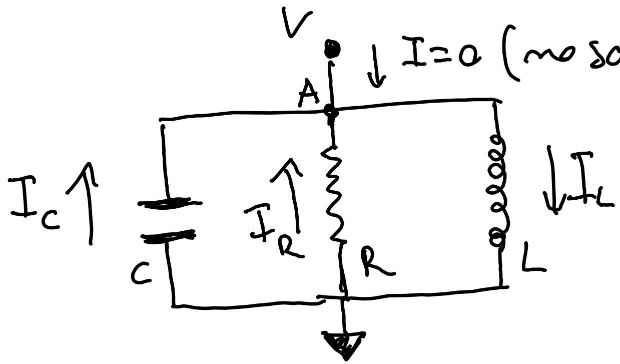
$$\hat{\Phi} = i\sqrt{\frac{\hbar Z_c}{2}}\ (\hat{a}^+ - \hat{a}) \longrightarrow$$ relate to magnetic field stored in inductor

or $$\hat{V} = \sqrt{\frac{\hbar \omega}{2 C}}\ (\hat{a}^+ + \hat{a})$$

$$\hat{I} = i\sqrt{\frac{\hbar \omega}{2 L}}\ (\hat{a}^+ - \hat{a})$$

with $\omega = \frac{1}{\sqrt{LC}}$ and $V = \frac{Q}{C}$ and $I = \Phi / L$

Dissipation in the Harmonic Oscillator



with

- current through resistor
  $I_R = V/R$
- displacement current
  $I_C = \dot{Q}_C = C\dot{V}$
- voltage across inductor
  $V = -L\dot{I}_L$

- Kirchoff law at point A

$$I_L = I_R + I_C + I$$

$$I_L - C\dot{V} - V/R = 0 \quad \text{(same voltage at A)}$$

$$I_L - C(-L\ddot{I}_L) + \frac{L}{R}\dot{I}_L = \frac{1}{LC} I_L + \ddot{I}_L + \frac{1}{RC}\dot{I}_L = 0$$

differential equation for current through inductor

- Solution $I_L(t) = I_L(0)\, e^{\lambda t}$ with $\lambda_{1,2} = \frac{1}{2LC}\left(-\frac{L}{R} \pm \sqrt{\left(\frac{L}{R}\right)^2 - 4LC}\right)$

Energy Decay Rate

- underdamped oscillator $(4LC \gg L/R)$

$$\lambda_{1,2} = -\frac{1}{2RC} \pm i\frac{1}{\sqrt{LC}} = -\alpha \pm i\omega$$

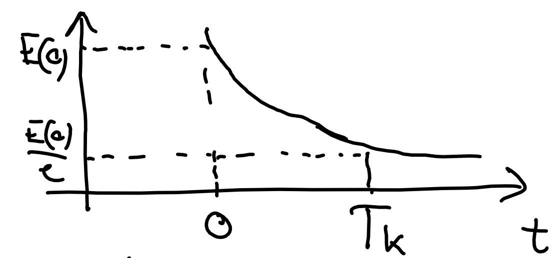
with $\alpha = \frac{1}{2RC} = \frac{1}{\tau}$ amplitude decay constant

$\tau = 2RC$ " " time

$\omega = 1/\sqrt{LC}$ oscillator frequency

- energy decay rate

$$E \propto \frac{1}{2} L I_L^2 \propto e^{-\frac{1}{RC}t}$$

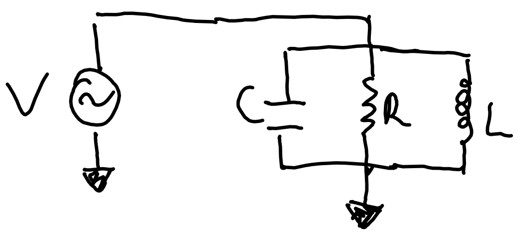


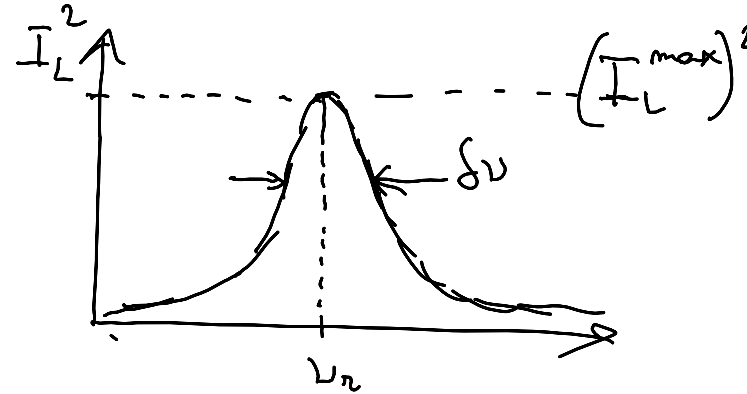
with $k = \frac{1}{RC}$ energy decay rate

$T_k = RC$ " " time

Spectral Response of Damped Harmonic Oscillator

- driven damped oscillator



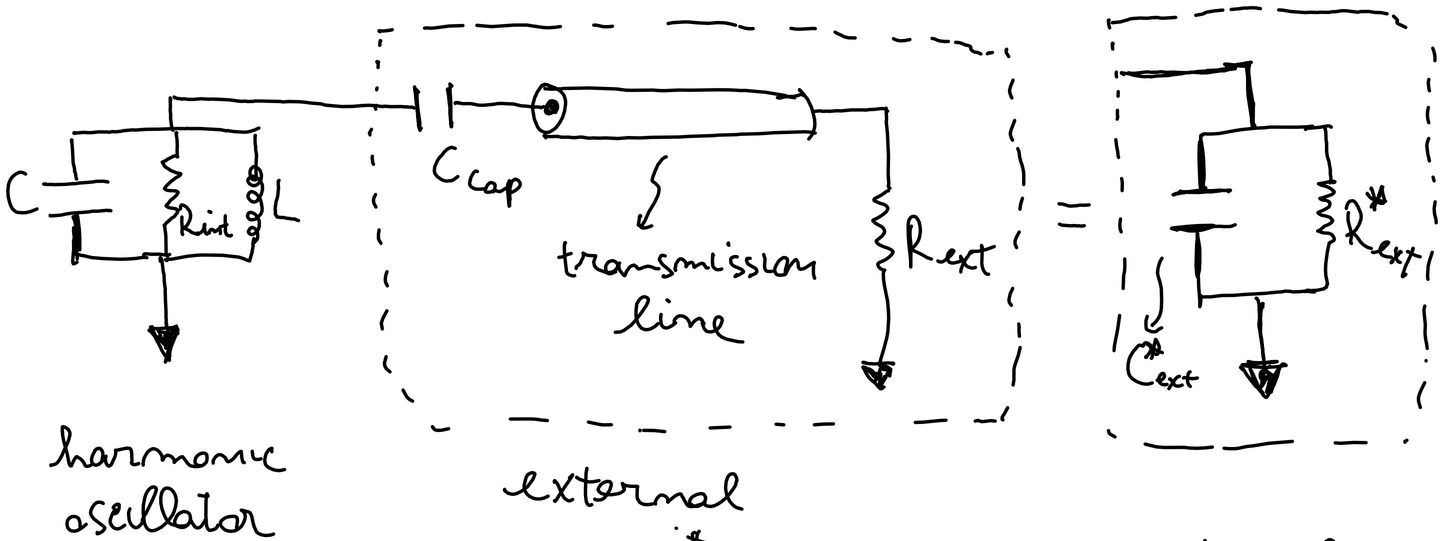


Lorentian line shape

$$I_L^2(\nu) = (I_L^{max})^2 = \frac{\delta\nu/\pi}{(\nu-\nu_r)^2+\delta\nu^2}$$

with $\delta\nu$: full width of line at half max

Internal and External Dissipation



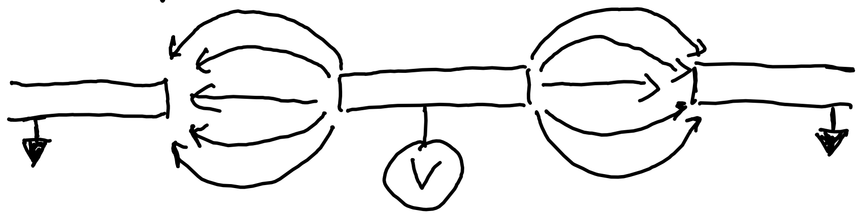
- total effective resistance $\frac{1}{R_{tot}} = \frac{1}{R_{int}} + \frac{1}{R^*_{ext}}$

- total effective capacitance $C_{tot} = C_{int} + C^*_{ext}$ frequency shift due to external circuit

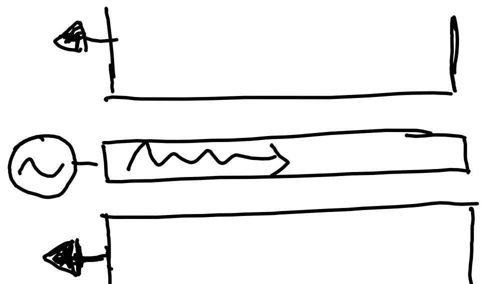
- energy decay time of combined system $T_K = R_{tot}\, C_{tot}$

Transmission line resonator

Field distrubution of the transmission line



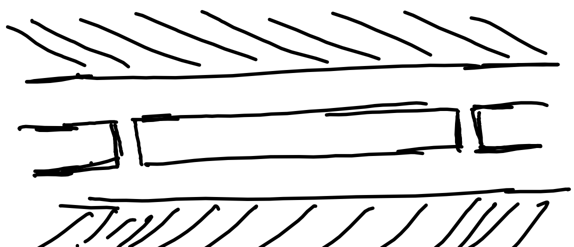
Wave propagation in a transmission line



$$E = E_0 e^{i(kx-\omega t)}$$

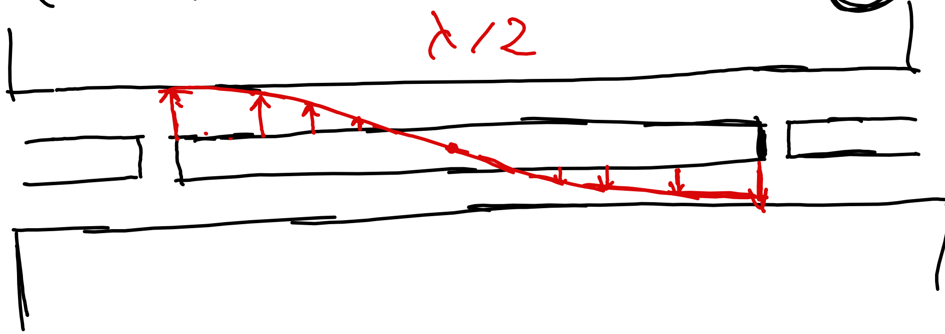
$$\omega = c_{eff} k = \frac{c}{\sqrt{\varepsilon}} \frac{2\pi}{\lambda}$$

Electric field at the boundary



Capacitor as a mirror. defining nodes for the current, so antinodes (maximum) for the voltage.

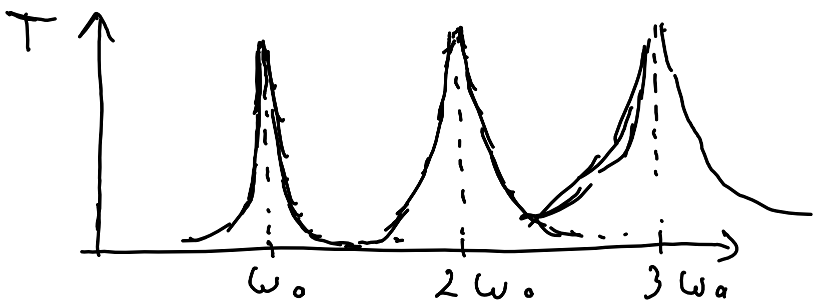
Fundamental mode



Resonance frequency?

$$\left.\begin{aligned} \omega &= c_{eff}\,\frac{2\pi}{\lambda} \\ l &= \frac{\lambda}{2}\,n \end{aligned}\right\}\ \omega = c_{eff}\,\frac{2\pi}{2l} = \frac{\pi c_{eff}}{l}\,n$$

Excitation spectrum:



"Harmonic oscillator"